Game Theory Explorer - Software for the Applied Game Theorist

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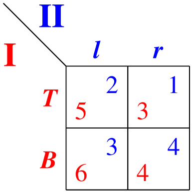
Overview

Explain and demonstrate GTE (Game Theory Explorer),

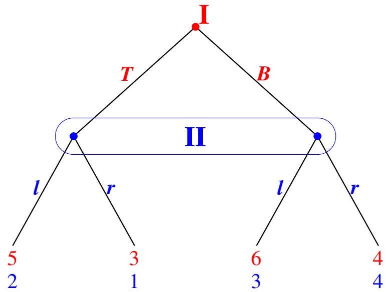
open-source software, **under development**, for

creating and analyzing non-cooperative **games**

in strategic form:



and extensive form:



Intended users

Applied game theorists:

- experimental economists (analyze game before running experiment)
- game-theoretic modelers in biology, political science, . . .
- in general: non-experts in equilibrium analysis

$\Rightarrow$ design goal: **ease of use**

Researchers in game theory:

- testing conjectures about equilibria
- as contributors: designers of game theory algorithms

Educators:

- interactive tool to explain solution concepts and algorithms

History: Gambit

GTE now part of the **Gambit** open-source software development,
`http://www.gambit-project.org`

2011, 2012, 2014, and 2016 supported by Google Summer of Code (GSoC)

Gambit software started ∼1990 with **Richard McKelvey** (Caltech) to analyze games for **experiments**, developed since 1994 with **Andy McLennan** into C++ package, since 2001 maintained by **Ted Turocy** (UEA, Norwich, UK).

- Gambit must be **installed** on PC/Mac/Linux, with GUI (graphical user interface) using platform-independent wxWidgets
- has collection of algorithms for computing Nash equilibria
- offers **scripting language**, now developed using Python

Features of GTE

GTE independent **browser-based** development:

- no software installation needed, low barrier to entry
- nicer GUI than Gambit
- export to graphical formats

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Other Contributors: David Avis (**lrs**), Mark Egesdal (2011), Alfonso Gomez-Jordana, Martin Prause, Christian Pelissier (**GSoC 2011, 2012, 2014**), Cesar de la Vega (2015), Harkirat Singh, Jaume Vives, Amelie Heliou (**GSoC 2016**)

Example of a game

2 × **2** game in strategic form:

I \ II	*l*	*r*
T	5, 2	3, 1
B	6, 3	4, 4

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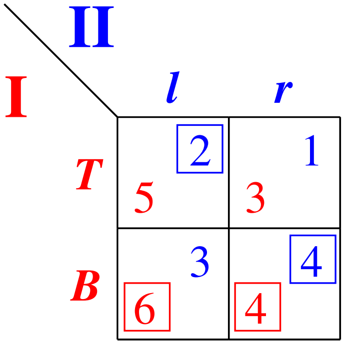

with pure best responses

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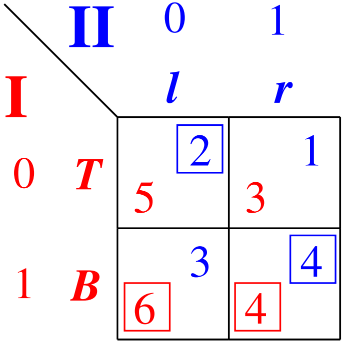


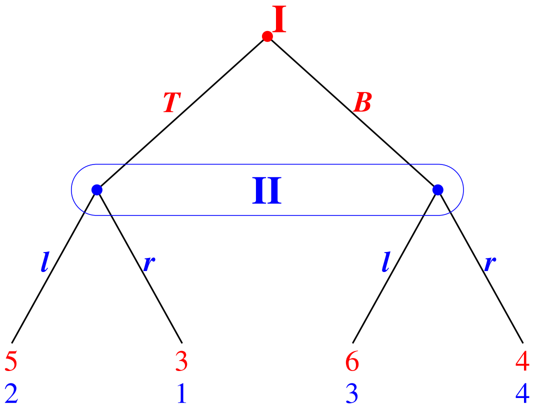
with pure best responses

and equilibrium probabilities

Extensive (= tree) form of the game

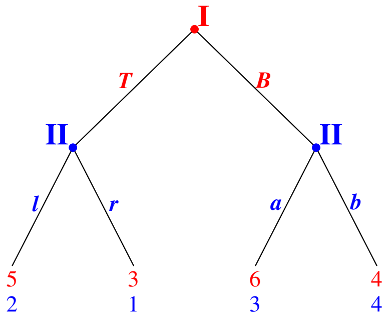
Players move sequentially,

information sets show **lack of information** about game state:



Commitment (leadership) game

Changed game when player **I** can commit:



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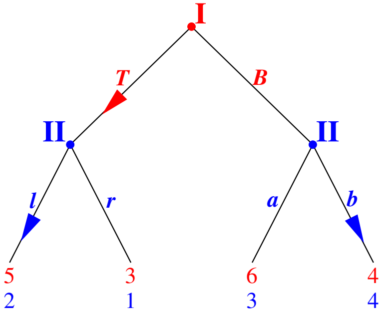

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

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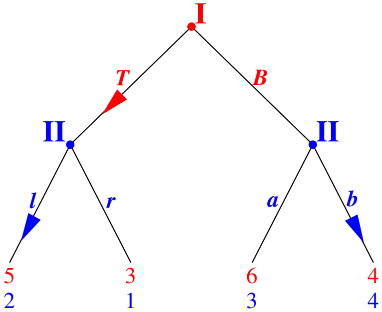

I \ II	*l–a*	*l–b*	*r–a*	*r–b*
T	2 5	2 5	1 3	1 3
B	3 6	4 4	3 6	4 4

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

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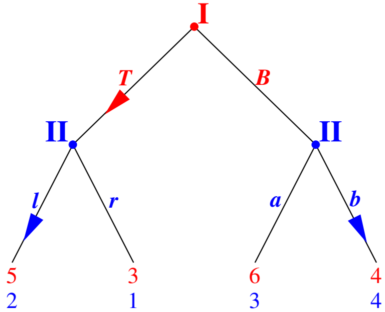

I \ II	*l–a*	*l–b*	*r–a*	*r–b*
T	5, 2	5, 2	3, 1	3, 1
B	6, 3	4, 4	6, 3	4, 4

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

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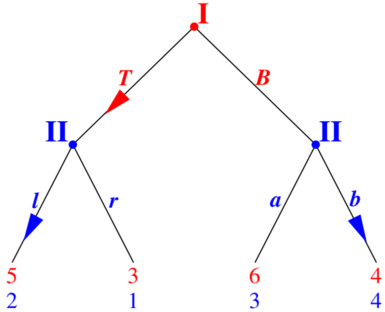

	I \ II	0 *l–a*	1 *l–b*	0 *r–a*	0 *r–b*
1	*T*	5, 2	5, 2	3, 1	3, 1
0	*B*	6, 3	4, 4	6, 3	4, 4

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

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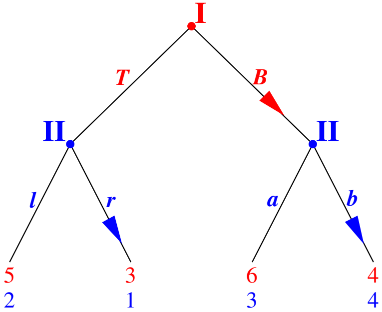


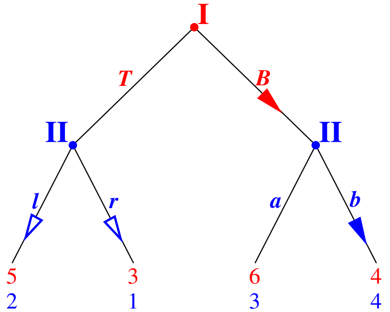
I \ II		0 *l–a*	0 *l–b*	0 *r–a*	1 *r–b*
0	*T*	5, 2	5, 2	3, 1	3, 1
1	*B*	6, 3	4, 4	6, 3	4, 4

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

Other equilibria: **(*B*, *r*-*b*)**

Commitment (leadership) game

Changed game when player **I** can commit:



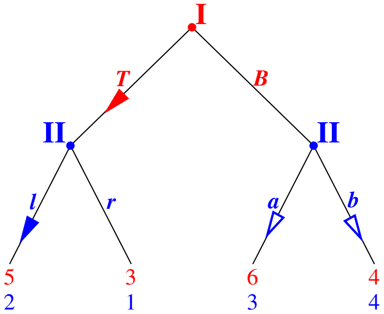
		II: 0	1/2	0	1/2
I		*l–a*	*l–b*	*r–a*	*r–b*
0	*T*	5, 2	5, 2	3, 1	3, 1
1	*B*	6, 3	4, 4	6, 3	4, 4

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

Other equilibria: **(*B*, *r*-*b*), (*B*, $\frac{1}{2}$*l*-*b* $\frac{1}{2}$*r*-*b*)**

Commitment (leadership) game

Changed game when player **I** can commit:



I \ II		1/2	1/2	0	0
		l–a	*l–b*	*r–a*	*r–b*
1	*T*	5, 2	5, 2	3, 1	3, 1
0	*B*	6, 3	4, 4	6, 3	4, 4

Subgame perfect equilibrium: **(*T*, *l*-*b*)**

Other equilibria: **(*B*, *r*-*b*)**, **(*B*, $\frac{1}{2}$*l*-*b* $\frac{1}{2}$*r*-*b*)**, **(*T*, $\frac{1}{2}$*l*-*a* $\frac{1}{2}$*l*-*b*)**

GTE output for the commitment game

```
2 x 4 Payoff player 1         2 x 4 Payoff player 2
  l-a l-b r-a r-b               l-a l-b r-a r-b
T   5   5   3   3           T   2   2   1   1
B   6   4   6   4           B   3   4   3   4

EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:
EE 1 P1: (1) 0 1 EP= 4 P2: (1)   0 1/2 0 1/2 EP= 4
EE 2 P1: (1) 0 1 EP= 4 P2: (2)   0   0 0   1 EP= 4
EE 3 P1: (2) 1 0 EP= 5 P2: (3)   0   1 0   0 EP= 2
EE 4 P1: (2) 1 0 EP= 5 P2: (4) 1/2 1/2 0   0 EP= 2

Connected component 1:
{1}  x  {1, 2}

Connected component 2:
{2}  x  {3, 4}
```

Demonstration of GTE

Preceding games:

- **2** $\times$ **2** game in strategic form
- extensive form of that game
- commitment game, extensive and strategic form

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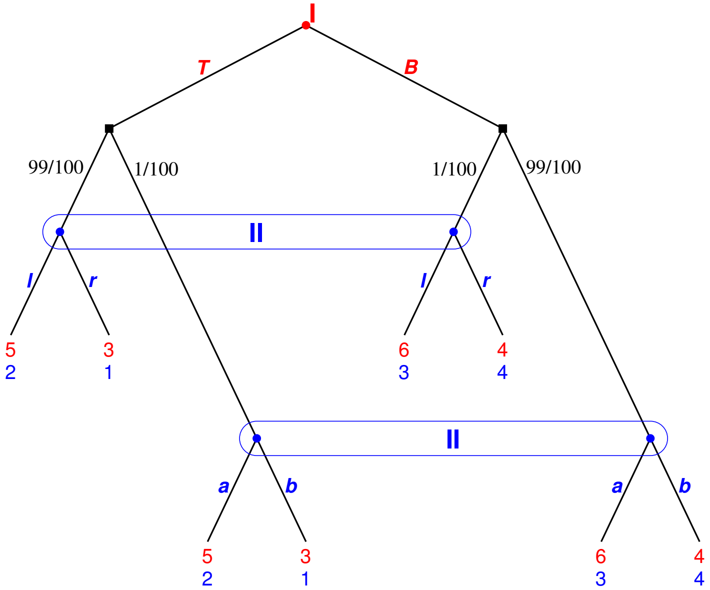
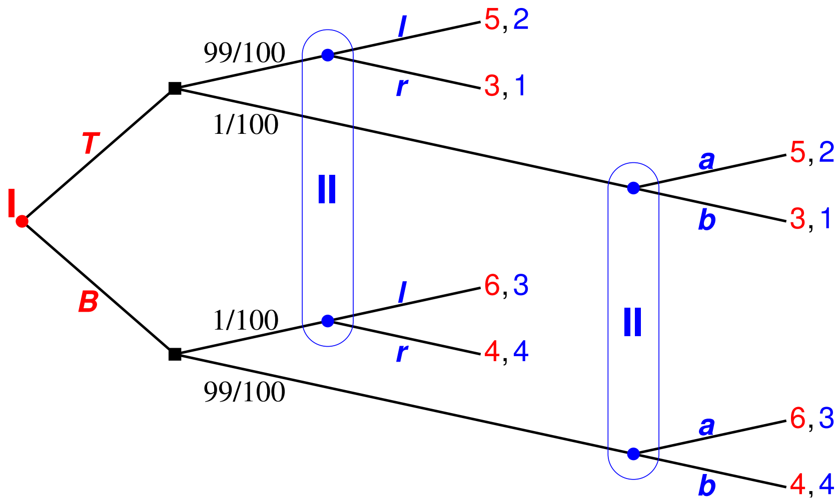
Next: create from scratch a more complicated extensive game

- imperfectly observable commitment

Game with imperfectly observable commitment



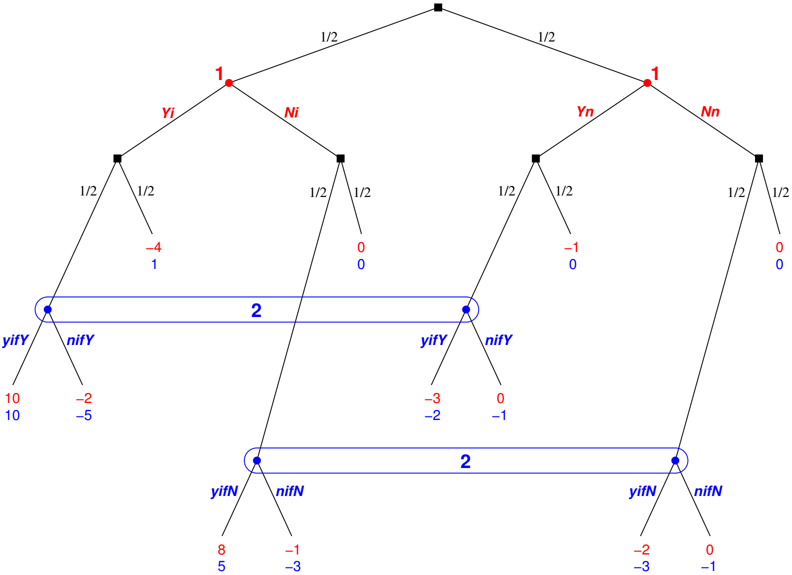
Game tree drawn left to right



GTE output for imperfectly observable commitment

```
2 x 4 Payoff player 1            2 x 4 Payoff player 2

  l-a    l-b    r-a r-b              l-a     l-b     r-a r-b
T   5 249/50 151/50   3          T   2 199/100 101/100   1
B   6 201/50 299/50   4          B   3 399/100 301/100   4

EE = Extreme Equilibrium, EP = Expected Payoffs

Decimal:
EE 1 P1: (1) 0.01 0.99 EP= 4.0102 P2: (1)      0 0.5102 0 0.4898 EP= 3.97
EE 2 P1: (2)    0  1.0 EP=    4.0 P2: (2)      0      0 0    1.0 EP=  4.0
EE 3 P1: (3) 0.99 0.01 EP= 4.9898 P2: (3) 0.4898 0.5102 0      0 EP= 2.01

Connected component 1:
{1}  x  {1}

Connected component 2:
{2}  x  {2}

Connected component 3:
{3}  x  {3}
```

More complicated signaling game, 5 equilibria



Some more strategic-form games

For studying more complicated games:

generate game matrices as text files, copy and paste into strategic-form input.

Future extension:

Automatic generation via command lines or “worksheets” for scripting, connection with Python and Gambit

GTE software architecture

Client (your computer with a browser):

- GUI: JavaScript (Flash's variant called ActionScript)
- store and load game described in XML format
- export to graphic formats (.png or XFIG $\rightarrow$ EPS, PDF)
- for algorithm: send XML game description to server

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Server (hosting client program and equilibrium solvers):

- converts XML to Java data structure (similar to GUI)
- solution algorithms as binaries (e.g. C program **lrs**), send results as text back to client

High usage of computation resources

Finding all equilibria takes exponential time

$\Rightarrow$ for large games, server should run on your computer, not a public one

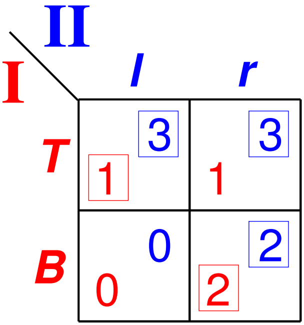
achieved by local server installation (“Jetty”), requires installation, but offers same user interface.

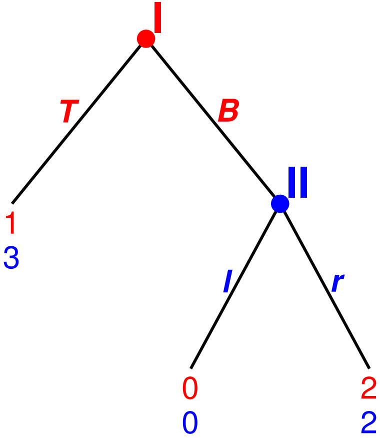
Algorithm: Finding all equilibria

For two-player games in strategic form, all Nash equilibria can be found as follows:

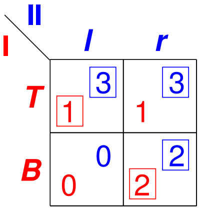
- payoffs define inequalities for “best response polyhedra”
- compute **all vertices** of these polyhedra (using **lrs** by David Avis, requires arbitrary precision integers)
- match vertices for **complementarity** (LCP)
- find maximal **cliques** of matching vertices for equilibrium **components**

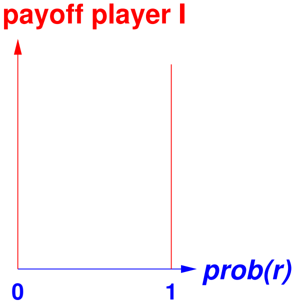
Example





Best response polyhedron of player I





Best response polyhedron of player I

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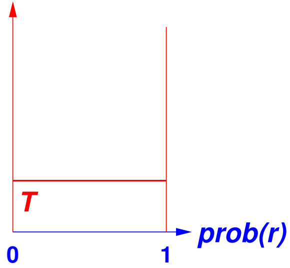
payoff player I



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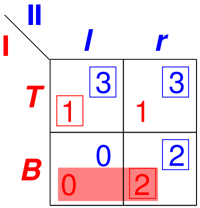

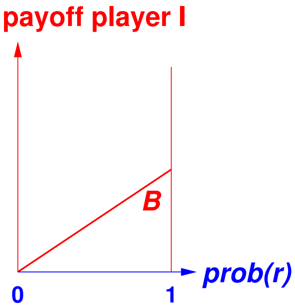

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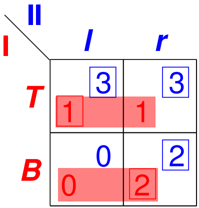
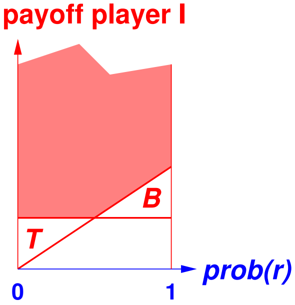
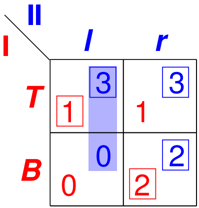
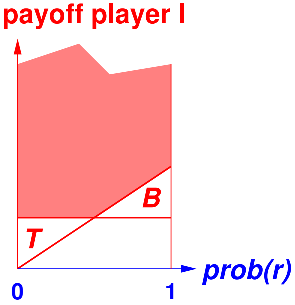
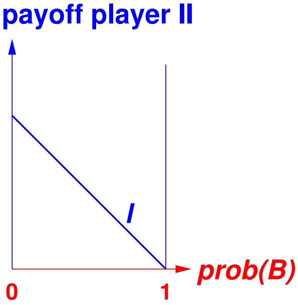




Best response polyhedron of player II







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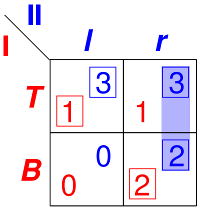

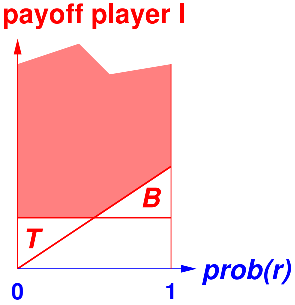

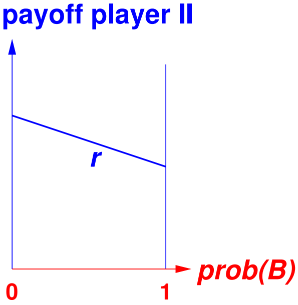

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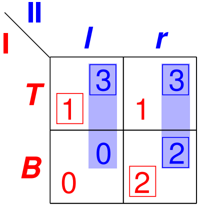
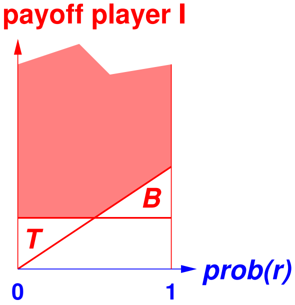
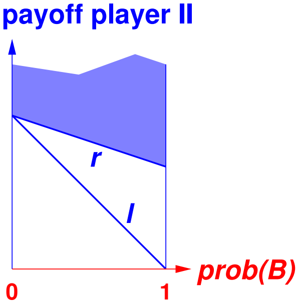
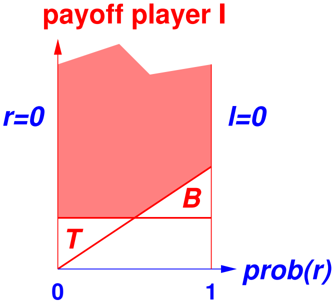


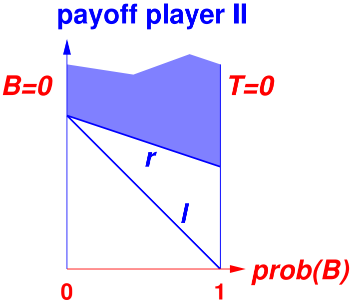




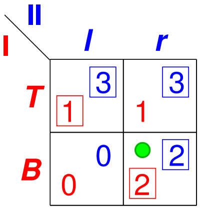
Label with best responses and unplayed strategies

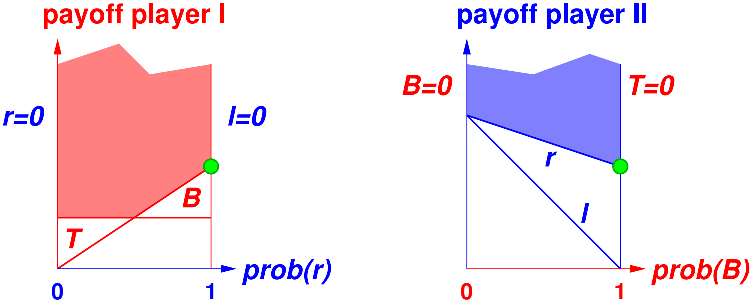
I \ II	*l*	*r*
T	1, 3	1, 3
B	0, 0	2, 2





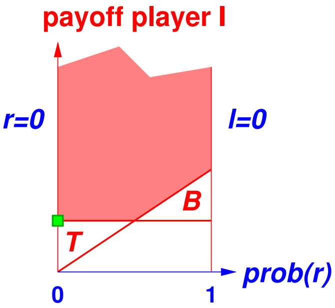
Equilibrium = **all** labels *T*, *B*, *l*, *r* present

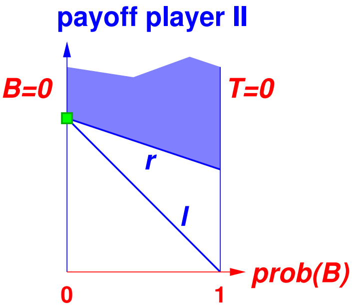




Equilibrium with multiple label *r* (degeneracy)

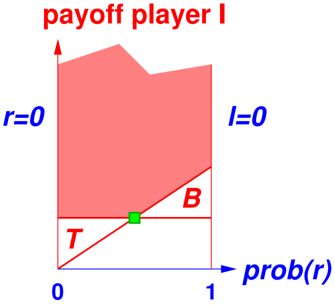
I \ II	*l*	*r*
T	1, 3	1, 3
B	0, 0	2, 2

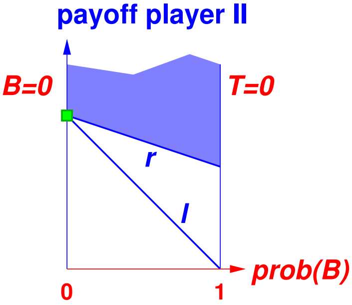




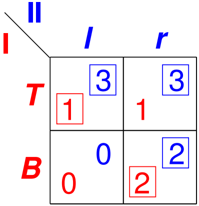
Equilibrium with multiple label *B* (degeneracy)

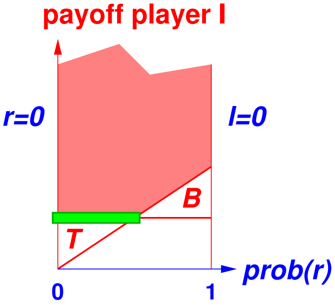
I \ II	*l*	*r*
T	1, 3	1, 3
B	0, 0	2, 2

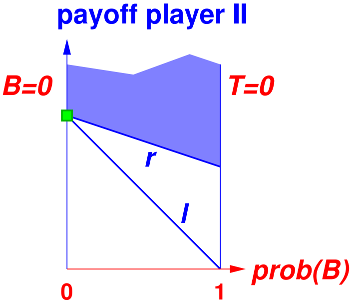




⇒ equilibrium component with labels ***T*** and ***B***, ***l***, ***r***







Equilibrium components via cliques

In degenerate games (= vertices with zero basic variables, occur for game trees), get convex combinations of "exchangeable" equilibria. Recognized as **cliques** of matching vertex pairs:

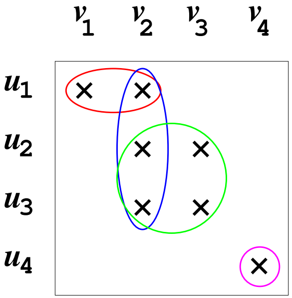
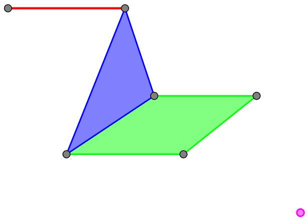


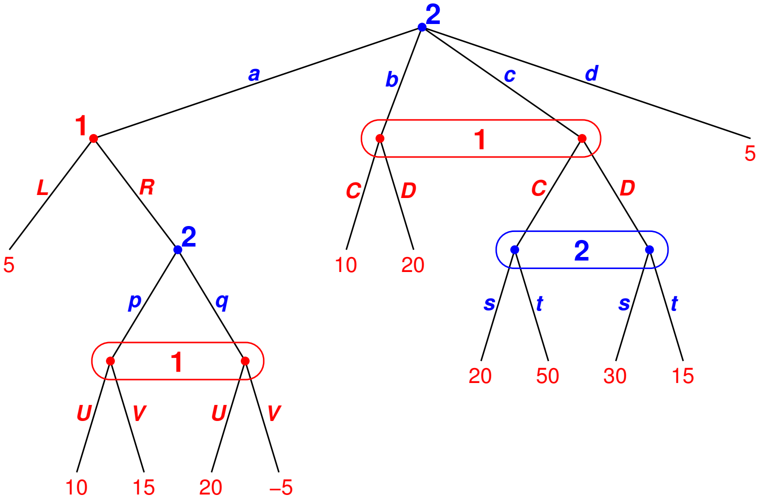
table of extreme equilibria



geometry

Algorithm: Sequence form for game trees

Example of game tree:



Exponentially large strategic form

Strategy of a player:
specifies a move for every information set of that player
(except for unspecified moves $*$ at unreachable information sets)

$\Rightarrow$ **exponential** number of strategies

	$ap*$	$aq*$	$b{*}{*}$	$c{*}s$	$c{*}t$	$d{*}{*}$
$L{*}C$	5	5	10	20	50	5
$L{*}D$	5	5	20	30	15	5
RUC	10	20	10	20	50	5
RUD	10	20	20	30	15	5
RVC	15	–5	10	20	50	5
RVD	15	–5	20	30	15	5

Sequences instead of strategies

Sequence specifies moves only along **path** in game tree

$\Rightarrow$ **linear** number of sequences, sparse payoff matrix $\boldsymbol{A}$

	$\emptyset$	*a*	*b*	*c*	*d*	*ap*	*aq*	*cs*	*ct*
$\emptyset$					5				
L		5							
R									
RU						10	20		
RV						15	–5		
C			10					20	50
D			20					30	15

Expected payoff $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}$, play rows with $\boldsymbol{x} \geq \mathbf{0}$ subject to $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{e}$,

play columns with $\boldsymbol{y} \geq \mathbf{0}$ subject to $\boldsymbol{F}\boldsymbol{y} = \boldsymbol{f}$.

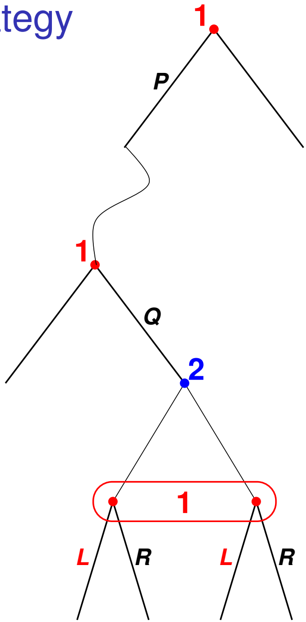
Play as behavior strategy

Given: $\boldsymbol{x} \geq \mathbf{0}$ with $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{e}$.

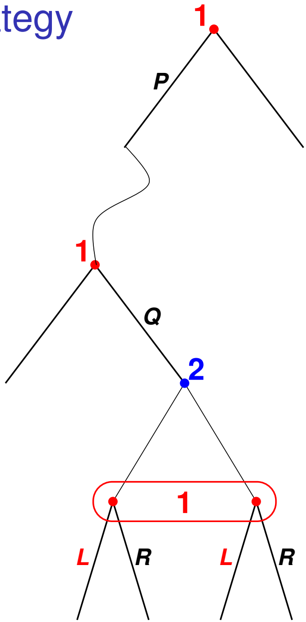
Move $\boldsymbol{L}$ is last move of **unique** sequence, say $\boldsymbol{PQL}$, where one row of $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{e}$ says

$$\boldsymbol{x}_{\boldsymbol{PQL}} + \boldsymbol{x}_{\boldsymbol{PQR}} = \boldsymbol{x}_{\boldsymbol{PQ}}$$

$$\Rightarrow \quad \text{behavior-probability}(\boldsymbol{L}) = \frac{\boldsymbol{x}_{\boldsymbol{PQL}}}{\boldsymbol{x}_{\boldsymbol{PQ}}}$$



Play as behavior strategy



Given: $\boldsymbol{x} \geq \mathbf{0}$ with $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{e}$.

Move L is last move of **unique** sequence, say PQL, where one row of $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{e}$ says $x_{PQL} + x_{PQR} = x_{PQ}$

$$\Rightarrow \quad \text{behavior-probability}(L) = \frac{x_{PQL}}{x_{PQ}}$$

Required assumption of **perfect recall** [Kuhn 1953, Selten 1975]:
Each node in an information set is preceded by same sequence, here PQ, of the player's **own** earlier moves.

Linear-sized sequence form

Input: Two-person game tree with perfect recall.

Theorem [Romanovskii 1962, von Stengel 1996]

The equilibria of a **zero-sum** game are the solutions to a Linear Program (LP) of **linear** size in the size of the game tree.

Linear-sized sequence form

Input: Two-person game tree with perfect recall.

Theorem [Romanovskii 1962, von Stengel 1996]

The equilibria of a **zero-sum** game are the solutions to a Linear Program (LP) of **linear** size in the size of the game tree.

Theorem [Koller/Megiddo/von Stengel 1996, von Stengel/Elzen/Talman 2002]

The equilibria of a **non-zero-sum** game are the solutions to a Linear Complementarity Problem (LCP) of linear size.

A sample equilibrium is found by **Lemke's algorithm**.

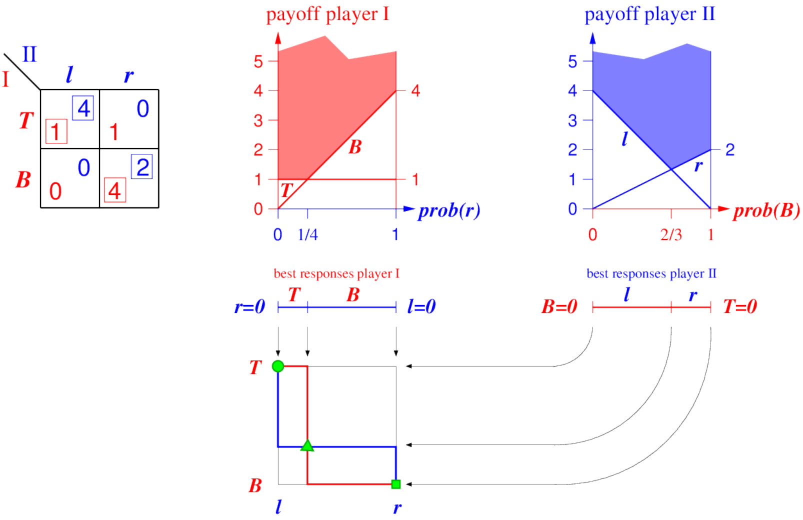
This algorithm mimicks the Harsanyi–Selten tracing procedure and finds a normal-form perfect equilibrium.

Google Summer of Code 2016

Three GSoC students currently working on:

- Improve and convert GUI to **pure JavaScript**
- Advanced **game tree layout** e.g. drawing information sets in games without time structure
- **Educational features** (example next)

Example of educational feature



Planned Extensions

Further solution algorithms:

- **EEE** [Audet/Hansen/Jaumard/Savard 2001]
- Path-following algorithms (Lemke-Howson, variants of Lemke)
- ***n***-player games: simplicial subdivision, polynomial inequalities

Scripting features:

- connect with Gambit and Python
- database of reproducible computational experiments

Educational features:

- teaching algorithms interactively

Summary

GTE – Game theory explorer

- helps **create**, **draw**, and **analyze** game-theoretic models
- user-friendly, browser-based, low barriers to entry
- open-source, work in progress, welcomes contributors

`https://github.com/gambitproject/gte/`
`https://github.com/gambitproject/jsgte/`

Rahul Savani and Bernhard von Stengel (2016)
Game Theory Explorer – Software for the Applied Game Theorist
Computational Management Science 12, 5-33.